BACK

CHANGING ASSESSMENT CRITERIA IN A-LEVEL MATHEMATICS WITH *DERIVE*

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Increasing availability of portable personal technology capable of running symbolic manipulation software will have an effect not only on the way mathematics is taught but also on how it is assessed. This is particularly relevant in England, where final examinations are set and marked by independent external bodies. This paper summarises the main aspects of the current debate, and gives examples of how the incorporation of computer algebra systems such as DERIVE may lead to more meaningful assessment.

INTRODUCTION

In England and Wales (the education system is slightly different in Scotland) school is compulsory until the age of 16. For those who wish to continue with academic (rather than vocational) studies, the next stage is the General Certificate of Education Advanced Level (commonly known as "A-level"). Students generally follow a two-year A-level course, and concentrate on three subjects. Good final grades in these subjects act as entry qualification to university. Thus A-levels have a similar role and status to the *Abitur* in Germany.

Unlike the *Abitur*, however, A-levels are assessed primarily on the basis of externally set and externally marked final written examinations. There exist around half a dozen Examination Boards, and each school can choose which Board it wishes to register with. While variations do exist between Boards in terms of syllabus content and assessment structure, they must all follow the basic guidelines of the Schools Curriculum Assessment Authority (SCAA). In mathematics, all Boards include an agreed common core of topics in their syllabus, and in the majority of cases between 80% and 100% of the final A-level grade is based on performance in two written end examinations set and marked by the Board.

The requirements of the Examination Boards obviously have a considerable influence on the way mathematics is taught in schools. Schools tend to be judged on the grades achieved by their students, and teachers of A-level mathematics generally perceive their main task as preparing their students to perform well in the final exams. Thus the policy of the Boards towards computer algebra systems (CAS) such as *DERIVE* will have a significant impact on the position of CAS in mathematics education in England. The expectation that a hand-held pocket computer running *DERIVE* will be on the market by the end of 1995 at a much lower price than hitherto is currently driving the debate within SCAA and the Examination Boards about the use of such technology in exams and hence the way in which mathematics should be assessed in future to take this into account.

DERIVE IN A-LEVEL MATHEMATICS EXAMINATIONS ?

It has been estimated that around 65% of a typical A-level Pure Mathematics examination can be answered directly by using a CAS such as *DERIVE*. (A similar figure has been claimed for *Abitur* examinations in Germany.) Thus many of the techniques and algorithms for algebra and analysis which have traditionally formed a large part of an A-level course may be considered redundant and examination questions rendered trivial if CAS are allowed in exams in their current format. There is general consensus that a policy towards CAS should be formulated and appropriate changes made to the types of question on the A-level examination papers. Discussion is still taking place, and the Examination Boards are unwilling to commit themselves prematurely. However, four possible approaches can be identified:

(1) *CAS should not be allowed in examinations*. This will certainly be the case until 1998 at least, due to the long lead-time necessary to modify syllabi and prepare new exam papers. Furthermore, since there has already been a recent review of all A-level syllabi to bring them into line with the new (pre-16) National Curriculum in schools, one can perhaps understand an initial reluctance on the part of the mathematics subject groups to make more changes. The main argument, however, is one of fairness towards those students who simply cannot afford the cost (initially around £200) of a hand-held pocket computer running a CAS. It can be assumed that such machines will continue to be prohibited until costs fall to a "reasonable" level. Thereafter, it will be increasingly untenable to forbid CAS in examinations, especially since students will be increasingly likely to have experienced *DERIVE* or similar software during their course. Banning symbolic manipulators is only an option for the short term; the "breathing space" should be used by all those involved in teaching and assessing mathematics at A-level to acquaint themselves fully with the implications of computer algebra in the longer term.

(2) *Examinations should be set on the current syllabi in such a way that no advantage is gained by candidates who have use of a CAS*. This is the brief currently being worked to by an advisory group of SCAA. On the face of it, this could be seen as an invitation simply to outwit the computer. However "satisfying" we may find it to discover shortfalls in the CAS - for example, *DERIVE*'s soLve command does not provide general solutions to trigonometric equations - this cannot be allowed to form the basis of how A-level candidates are assessed. A similar proposal is to set questions in general terms which do not specify a function or a value for the CAS to work with (e.g. Given that (p,q) is a relative minimum point on the curve y = f(x), show that . . .), but too many abstract questions of this type may well alienate many candidates. Another way of implementing this approach would be to demand that "full working is shown" in order that candidates can demonstrate some understanding other than merely pressing computer buttons, but if this working were merely to take the form of carrying out algorithms the mastery of which CAS have made redundant, there would be a loss of credibility in the relevance of the examination. Assessment which is set in spite of, rather than in sympathy with, the technical support available to the candidate runs the danger of being perceived as increasingly unreal and irrelevant.

(3) *A differentiated scheme of assessment should be adopted.* This approach is similar to what happened for some years when pocket calculators became readily available in the 1970s. An examination could be made available in two options: one for candidates with CAS, and one for candidates without. This would allow schools the freedom to introduce CAS at their own pace, to allow for the time needed to retrain staff and build up appropriate computer resources. However, this approach would lay itself open to arguments about whether the two options were really of equal status in the eyes of employers and universities. A second type of differentiated assessment would be to set one exam paper where <u>all</u> calculating devices were prohibited, and a second paper where calculators and hand-held computers could be used. This would help to ensure that the "essential" concepts of number, algebra, calculus and graphs are still mastered, but allow the CAS as a tool to assist the solution of more complex problems. This has a certain amount of support among teachers who appreciate the power of CAS such as *DERIVE*, but who despair that students will end up relying on a computer to perform simple algebra just as many of them now rely on a calculator to carry out basic arithmetic.

(4) *Assessment criteria should be radically altered.* This approach is claimed to be the only realistic way in the longer term of ensuring that the opportunities provided by CAS for the study of mathematics are fully realised. Mathematical problem-solving is generally considered to be a process of three stages:

(i) formulate/model the given problem in mathematical terms;
(ii) apply appropriate methods to solve the mathematical problem;
(iii) interpret the solution and draw conclusions.

However, in traditional A-level Pure Mathematics examinations, emphasis is almost exclusively on the middle stage, with very little reference made to the initial formulation/justification of the problem or the interpretation of the final answer. Computer algebra systems can now cope with much of the routine manipulation and solution of the mathematics (the word "drudgery" is often used in this context!), freeing the student to concentrate on the appropriateness of the model to start with, and the validity of the solution. These skills should be assessed accordingly, either in a written examination or through a series of time-constrained tasks. Radically changing the assessment criteria in this way may be perceived as threatening by teachers who have so far coped successfully with the traditional style of examination. However, developments in this direction seem inevitable. It shall be claimed below that such changes need not be as difficult to implement as perhaps first thought.

A NEW APPROACH TO THE ASSESSMENT OF TRADITIONAL TOPICS WITH *DERIVE*

It is contended here that the growing availability of CAS will eventually result in assessment at A-level which places as much if not more emphasis on the formulation of a mathematical problem and the interpretation of the final answer than on the routine application of algebraic techniques which provide that answer. This is often taken by advocates of "real world" mathematics to justify an approach based on the modelling of applied mathematics problems, and indeed it is readily applicable to examination papers in Mechanics, Statistics or Decision Maths. However, it is also relevant to questions of the type found on traditional Pure Mathematics examination papers. The initial "modelling/formulation" stage would probably need to be more along the lines of a description (e.g. of a family of curves), a justification (e.g. for using a particular function) or an explanation (e.g. of a technique). The middle stage would allow the use of a CAS to obtain a solution otherwise found by the rote application of some algorithm. The final stage would be an interpretation of the solution, something rarely required of candidates hitherto. Such an approach would involve more writing, but it can be argued that this would ensure that the candidate is more fully aware of what the mathematics is about, rather than relying on the accurate following of barely understood techniques to gain marks.

The following question is taken from a recent A-level Pure Mathematics paper:

> Q1 The quadratic equation $x^2 + 6x + 1 = k(x^2 + 1)$ has equal roots.
> Find the possible values of the constant k.
>
> [AEB, June 1994]

This is a typical short question from the beginning of the paper. It is clear what is expected - a statement that the discriminant must be zero; the correct recall of the formula for the discriminant; the formulation of an equation in terms of k; the solution of the equation (itself a quadratic) for k. As such, this question typifies all that is ripe for change in this kind of A-level examination paper. The equation is given with no motivation or justification; the method of solution depends on the Pavlovian response "$b^2 - 4ac = 0$" to the statement of equal roots; there is no requirement to interpret the values of k obtained. Having completed the question, the student will end up none the wiser for the experience.

The question above can be solved fairly easily with DERIVE. The method adopted in the solution below is arguably "better" that the one based on the discriminant in that it relates more directly to the roots of the original equation:

#1:	$x^2 + 6*x + 1 = k*(x^2 + 1)$	User
#2:	$x = \dfrac{\mathrm{SQRT}(2*(k + 4) - k^2) + 3}{k - 1}$	Solve(#1)
#3:	$x = \dfrac{\mathrm{SQRT}(2*(k + 4) - k^2) - 3}{1 - k}$	Solve(#1)
#4:	$\dfrac{\mathrm{SQRT}(2*(k + 4) - k^2) + 3}{k - 1} = \dfrac{\mathrm{SQRT}(2*(k + 4) - k^2) - 3}{1 - k}$	User
#5:	$k = -2$	Solve(#4)
#6:	$k = 4$	Solve(#4)

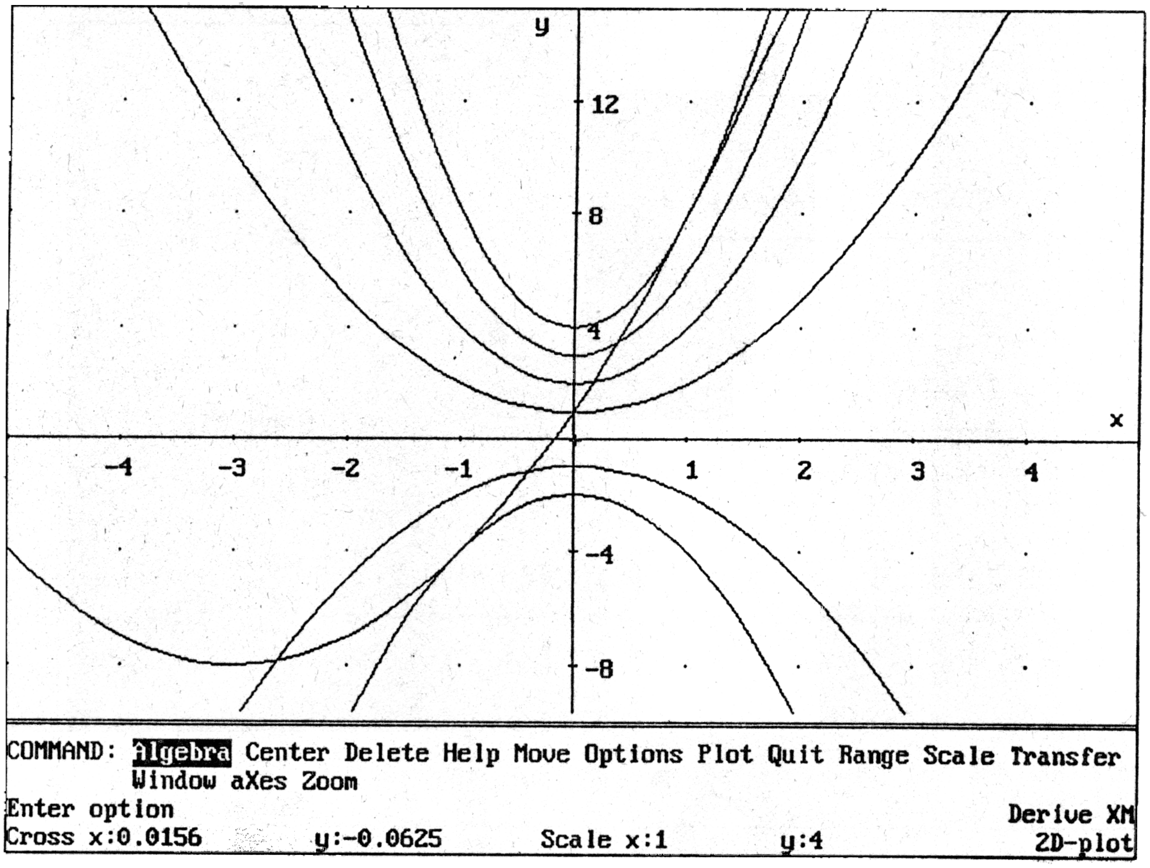
Although *DERIVE* can provide the solution to this question so routinely, the properties of quadratic functions and equations remain an essential part of A-level mathematics. What is needed is a way to assess the student's understanding of these properties, using *DERIVE* as a tool to support that understanding. A clue to one possible approach is given in the way the question above is formulated. The right-hand-side of the given equation is in fact a family of parabolas whose shape varies with k. The student should be able to explain the effect of varying k, and visualise the ways in which a fixed parabola and a variable parabola can intersect. This leads to a proposed re-formulation of the original question:

Q1* Sketch the graph $y = x^2 + 1$.
Explain the effect of the constant k on the shape of the graph
$$y = k(x^2 + 1).$$
Sketch on the same axes the family of curves given by
$$y = k(x^2 + 1) \quad \text{for} \quad k = -2[1]4.$$

Find the values of k for which the quadratic equation
$$x^2 + 6x + 1 = k(x^2 + 1) \qquad \text{has equal roots.}$$

Explain, with diagrams as appropriate, what this tells you about the graphs $y = x^2 + 6x + 1$ and $y = k(x^2 + 1)$ for these values of k.

Thus the original question remains, but the emphasis is on providing a context and hence motivation for the question to start with, and on interpreting the solution based on the corresponding graphs.



DERIVE's algebra and graph plotting facilities support the work throughout. It would be acceptable for the student to sketch the graphs by copying clearly what appears in the *DERIVE* plot window, provided all the important features of the curves are included. The student will surely feel some achievement when obtaining a graph similar to the one shown above.

The same A-level Pure Mathematics paper also contained the following question:

> Q2 Given that $|x| < \frac{1}{4}$, write down the binomial expansion of $(1-4x)^{-\frac{1}{2}}$ in ascending powers of x up to the term in x^3 .
>
> Hence obtain the coefficient of x^3 in the expansion of $\dfrac{1-3x}{\sqrt{1-4x}}$.
>
> [AEB, June 1994]

Binomial expansions are common at A-level, but the underlying powerful mathematical theory of polynomial approximation is rarely appreciated by the student, who sees such a question as a way to gain a few marks by "churning out" the series according to the standard pattern. No opportunity is given to reflect on what has been achieved, or to check or verify the final answer. This question is trivial using *DERIVE*, with the second part no more difficult than the first:

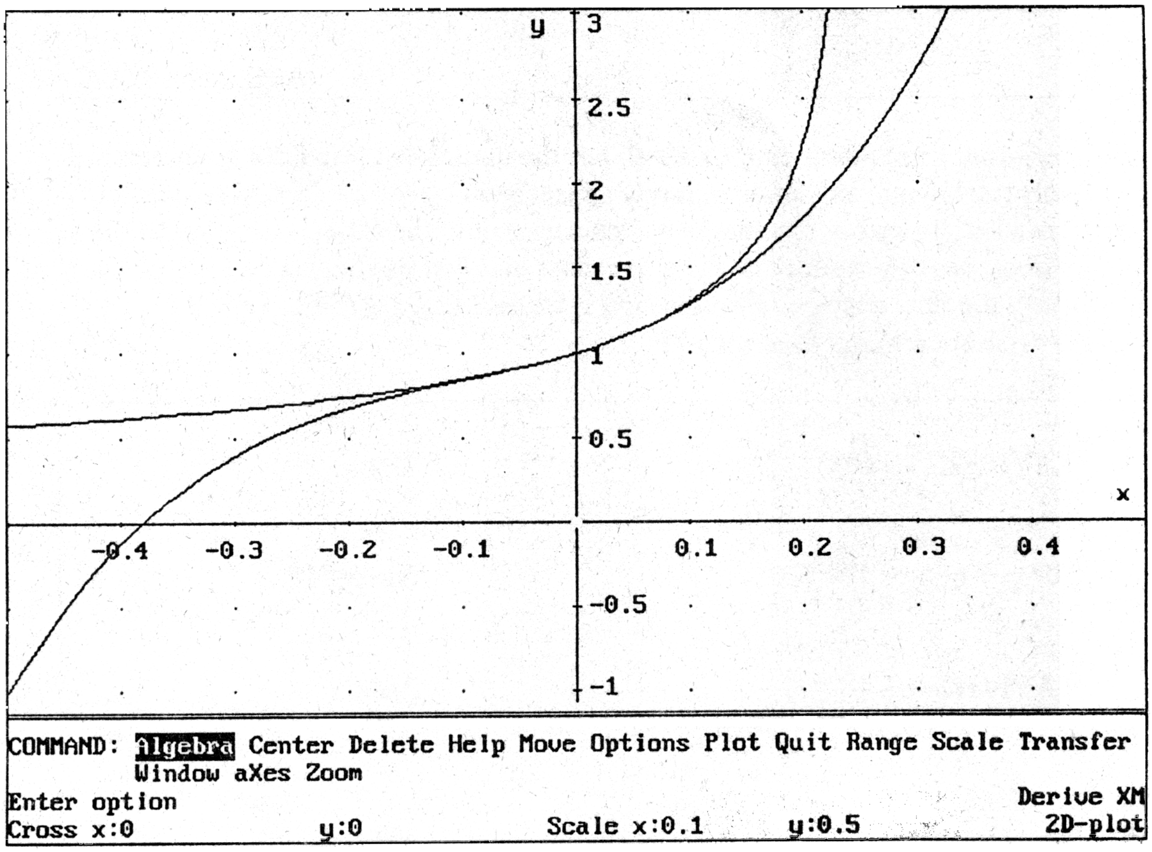
```
                        - 1/2
#1:  TAYLOR((1 - 4*x)       , x, 3)                       User

          3       2
#2:  20*x  + 6*x  + 2*x + 1                               Simp(#1)

               /   1 - 3*x         \
#3:  TAYLOR|------------------ , x, 3 |                   User
               \ SQRT(1 - 4*x)     /

        3
#4:  2*x  - x + 1                                         Simp(#3)
```

A possible re-formulation of this question, which allows the derivation of the series to be found using DERIVE and places more emphasis on interpreting the practicality of the final result, is given below:

> Q2* What is meant by the "domain" of a function? State, with a reason, the domain of the function $f(x) = (1-4x)^{-\frac{1}{2}}$
>
> The function *f(x)* is to be approximated by a cubic polynomial *p(x)*. Find *p(x)*. For what range of values of *x* is this polynomial approximation valid?
>
> Sketch on the same axes the graphs $y = f(x)$ and $y = p(x)$ for $-0.4<x<0.4$. Explain briefly how your graphs confirm the valid range of *x* you found above.
>
> *(cont'd ...*

By reference to your graphs, explain what errors in (a) function values, (b) first derivative values, (c) definite integral values you would expect to encounter if you used *p(x)* to approximate *f(x)*.

It has already been shown how DERIVE's algebra capabilities make short work of obtaining the cubic approximation, allowing the candidate time to produce a graphical interpretation of the result (see below) and to appreciate more fully the validity of the polynomial approximation.



CONCLUSION

The increasing availability and affordability of hand-held computers capable of running a CAS such as *DERIVE* will not only have an impact on the way mathematics is taught and learnt, but also on how it is assessed. If the A-level examinations are to retain their status, the Examining Boards must be seen to be working towards a type of assessment which requires candidates to have studied and appreciated the essential topics of algebra, calculus and so on, but to demonstrate their understanding in qualitatively new ways.

Two examples are shown of recent A-level Pure Mathematics questions which in their original form rely only on the routine and error-free application by hand of standard algorithms. It is to be feared that students prepare for the examination by merely practising such algorithms repeatedly, devoid of any need to appreciate what the underlying mathematics actually is. Alternatives to these two questions are proposed, where the calculations originally required may be delegated to *DERIVE*, allowing the student to concentrate on the formulation of the problem and the interpretation of the solution.

The intention of this paper is to demonstrate that the standards of the traditional Pure Mathematics A-level need not be compromised by allowing CAS into the examination room. The same topic areas as hitherto can be covered in the syllabus, and the same categories of functions, equations and graphs can be studies in the A-level course. Routine manipulation can still be carried out by hand whenever deemed appropriate in the classroom to consolidate certain topics. However, teachers and students should be aware that in the final examination, credit would be given primarily for understanding and interpretation (which will also be affected by SCAA's new "Quality of Language" guidelines).

Ultimately it will be the policy of the Examination Boards which determines the extent to which CAS will be allowed, or even encouraged, in A-level Mathematics. Until such a policy is formulated and implemented, however, teachers have time to re-appraise their own teaching and assessment in the light of the current debate surrounding the opportunities provided by CAS. One common thread is emerging, however: the routine practice of algebra skills will not be enough. The two subversive little words "So what?" should be uttered more often - by students and teachers - as a way of laying the path towards more meaningful learning and assessment of mathematics supported by software such as *DERIVE*.