BACK

Computer Algebra in Realistic Mathematics Education

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Technology and Realistic Mathematics Education

The domain specific instruction theory for realistic mathematics education has acquired a considerable following in The Netherlands during recent decades. According to this theory, realistic applications should play an important role in the learning process right from the start. Solution procedures are (re)constructed by the students themselves through using problems that have meaning in their reality. Some important characteristics of realistic mathematics education are:

- a variety of solution strategies
- a high degree of student input
- use of informal strategies and informal knowledge
- footholds for reflection
- stimulus for raising the level, for generalization and for formalization

For an extensive discussion on the theory of realistic mathematics education, see Freudenthal (1991) and Treffers (1987).

How do these general starting points of realistic mathematics education translate to the specific situation of implementing computer algebra software in math class? In the course of the graphics calculator research project (see Doorman et al, 1994[a]), this instruction theory became crystallized in five hypotheses, that we summarize as follows:

1 Realistic contexts

Mathematical models used in realistic applications often contain 'unsightly' numbers or formulas. For the sake of manageability, reality is sometimes twisted in order to assure a smooth problem. With the arrival of the technology devices, there is no longer any need to tarnish the realistic quality of the problem. The machine, after all, takes over the time-consuming technical work, leaving the student free to concentrate on the process of mathematization, the solution strategy and the drawing of reasonable conclusions.

2 Exploration

Thanks to its direct feedback, technology can offer opportunities for exploratory activities. Even during the initial phase of familiarization a problem can often already be investigated graphically. Inventory and classification activities can lead to discoveries that then, through reflection and generalization, result in interesting mathematical theorems. This contrasts with the traditional method, in which definitions and theorems often are stated at the beginning of the learning path in the expectation that insight will be acquired through repeated application.

3 Integration
Use of technology can contribute to the integration of the two classical components of mathematics: algebra and geometry. Operations with algebraic expressions via graphic (or geometrical) illustration of functions on the screen can by this means be performed in a great variety of ways. Algebraic laws and rules can thereby be discovered and then tested graphically. This can lead to a more visual form of algebra instruction.

4 Dynamics
The use of technology devices such as graphics calculators or computer algebra systems has a number of dynamic aspects.
Firstly, one can quickly and effectively follow the results of alterations in the problem or in the model. The influence of a given parameter in the formula can easily be made clear by using graphics.
Another dynamic aspect is the ability to trace a graph or curve with the cursor, while reading off the constantly changing coordinates on the screen. The speed of a movement can be made visible this way.
A third dynamic aspect is the ability to zoom in and out of the graph. This enables one to continually alter one's frame of reference from 'global' to 'local' and vice versa.

5 Flexibility
Due to the advent of technology, the repertoire of techniques and skills a student must master will markedly change. Freehand drawing of a graph based on a strictly prescribed analysis of functions - a much practiced skill up to now - will hardly matter. On the other hand, skills such as 'estimation', 'reading graphs' and 'successive approximation' will increase in significance.
All in all, we may expect that, due to the influence of technology, a shift in emphasis will occur away from 'rigid techniques' and towards a more flexible solution procedure.

In this article, we will look at two examples in order to find out whether these hypotheses also hold for the use of computer algebra in the teaching of mathematics at upper secondary level.

First Example: The population of China

In the Netherlands, we have two different mathematics subjects at upper secondary level: Mathematics A and Mathematics B. Math A is oriented towards further studies in the humanities and it focuses primarily on the application of mathematics. Formulas used in these applications are an attempt to represent reality and one can often suffice with approximate answers due to the model character inherent here (see De Lange, 1987). The main topics are statistics, (applied) calculus and the matrix models that are considered here.

This example fits in the Mathematics A philosophy, although part of the mathematical knowledge (especially concerning characteristic polynomials and eigenvalues) - goes beyond the level of Math A.

The population of China is growing quite fast. The authorities got worried about the immense number of people that would need food, work and health care in the nearby future, if the growth would continue at the same rate. They want to have a model that predicts the future development and that can help to estimate the effects of possible measures.

In 1982 there were 1008 millions of Chinese people. The distribution over the generations is shown below.

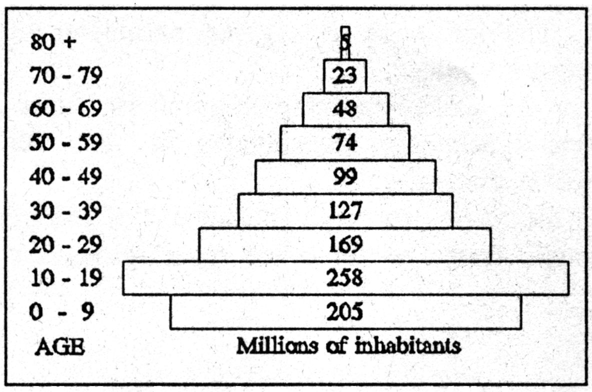

Fig. 1 The population distribution in 1982

Experts in the field of demography have estimated the fertility rates and the survival rates for each of the nine generations. These estimations are given in a population matrix or Leslie matrix:

$$\begin{pmatrix} 0 & .45 & .69 & .13 & 0 & 0 & 0 & 0 & 0 \\ .97 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & .993 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & .987 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & .981 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & .962 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & .907 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & .761 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & .51 & 0 \end{pmatrix}$$

Now, of course, we can use a computer algebra system, in this case Derive, to investigate the model. Let us look at the first session.

```
#2:  Precision := Approximate

#3:  DisplayFormat:=Compressed

               9
#4:  ADD(w):=  Σ  ELEMENT(w,i)
              i=1

#5:  v:=[205,258,169,127,99,74,48,23,5]

#6:  ADD(v)=1008

#7:  m·v=[249.22,198.85,256.194,166.802,124.587,95.238,67.118,36.528,11.73]

#8:  ADD(m·v)=1206.26

      20
#9:  m   ·v

#10: [1030.28,928.02,855.435,783.843,714.267,637.519,536.932,379.825,179.416]

           20
#11: ADD(m    v)=6045.55

#12: ▮
```

Fig. 2 Calculating the future development

Line 1 is not displayed. It contains the definition of the population matrix that is called m. The procedure 'ADD' is defined to calculate the sum of the vector elements to find out the total number of Chinese people.
From line 11 we conclude that, if the model is valid, the population growth up to about 6000 millions after twenty years, that is in 2002!

Figure 3 reveals that there will be relatively more older people in 2002 then in 1982, which is another dangerous development. Furthermore, we see that the growth rate, if it is a constant, seems to be about 9 percent each year.

```
#13: PrecisionDigits:=2

#14: DisplayFormat := Normal

      v·100
#15: ───────
      1008

#16: [20.3, 25.5, 16.7, 12.5, 9.8, 7.3, 4.7, 2.2, 0.49]

        20
      (m   ·v)·100
#17: ─────────────
        6045.5

#18: [17.0, 15.3, 14.1, 12.9, 11.8, 10.5, 8.8, 6.2, 2.9]

#19: ▮

#20: PrecisionDigits := 6

      ┌ 6045.5 ┐1/20
#21:  │────────│      = 1.0937
      └  1008  ┘
```

Fig. 3 Calculating the annual growth rate

In order to investigate the asymptotic behaviour of the model, we try to evaluate the eigenvalues of the Leslie matrix. Because the characteristic polynomial has degree nine, exact calculations are doomed to fail. But there are two alternatives: the graphical method (see figure 5) and the numerical approximations (see line 29 in figure 4). Both lead to a similar result: the biggest real eigenvalue is about 1.077, which is somewhat smaller than the 1.0937 we found before, but is still considerable.

#22: Precision := Exact

#23: CHARPOLY(m)

#24: $-\dfrac{w^5\cdot(10000000000\cdot w^4 - 4365000000\cdot w^2 - 6646149000\cdot w - 1235894751)}{10000000000}$

#25: $-\dfrac{10000000000\cdot w^4 - 4365000000\cdot w^2 - 6646149000\cdot w - 1235894751}{10000000000}$

#26: $-w^4 + \dfrac{873\cdot w^2}{2000} + \dfrac{6646149\cdot w}{10000000} + \dfrac{1235894751}{10000000000}$

#27: $-w^4 + 0.4365\cdot w^2 + 0.664614\cdot w + 0.123589$

#28: Precision := Approximate

#29: $w = 1.07707$

Fig. 4 Approximating the eigenvalue numerically

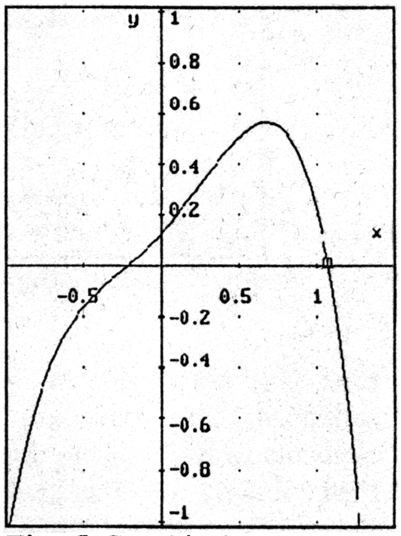

Fig. 5 Graphical approach

The Chinese authorities have tried to discourage their inhabitants to have more than one child. The assumed effect on the fertility rates will change the first row in the population matrix into (0, .41, .59, 0, 0, 0, 0, 0, 0). The new population matrix is called n, and we repeat the calculations for this new situation.

#32: Precision := Exact

#33: CHARPOLY(n)

#34: $-\dfrac{w^6\cdot(10000000\cdot w^3 - 3977000\cdot w - 5682939)}{10000000}$

#35: $-\dfrac{10000000\cdot w^3 - 3977000\cdot w - 5682939}{10000000}$

#36: $-w^3 + \dfrac{3977\cdot w}{10000} + \dfrac{5682939}{10000000}$

#37: $-w^3 + 0.3977\cdot w + 0.568293$

#38: Precision := Approximate

#39: $w = 0.986730$

Fig. 6 Approximated eigenvalue if measures are taken

Figure 6 shows that in this case the biggest real eigenvalue is approximated to 0.986730, which means that there will be negative exponential growth at the long end, which means that the dangerous positive exponential growth is avoided!
Of course, these results can be questioned. The main criticism probably is the fact that the fertility rates will not be constant over a longer period of time and that they cannot be estimated very accurate.
In spite of this, however, this example suggests how computer algebra can be valuable in the investigation of a realistic matrix model.

The second example: The rotation of a coin

This second example is developed for Mathematics B, which is the mathematics stream that prepares the student for subsequent exact studies. The curriculum for this subject is now under discussion. In this discussion, technology plays an important role. Therefore it is worthwhile developing student materials that open new horizons. The example, that stems from Doorman et al (1994[b]), deals with the following situation.

Two Dutch guilders lie next to one another. The left guilder is fixed. *P* is the point on the edge of the right guilder where it touches the left guilder. The right guilder is now going to roll around the edge of the left guilder without sliding. The question is, what path will *P* describe.

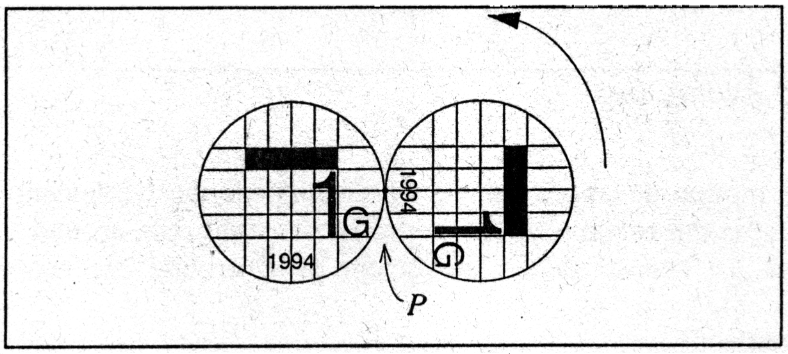

Fig. 7 The coins in the starting position

First, one can physically try it, and consider some starting questions. For example, can you tell where you will see '1994' when the moving coin is completely at the left of the fixed one?

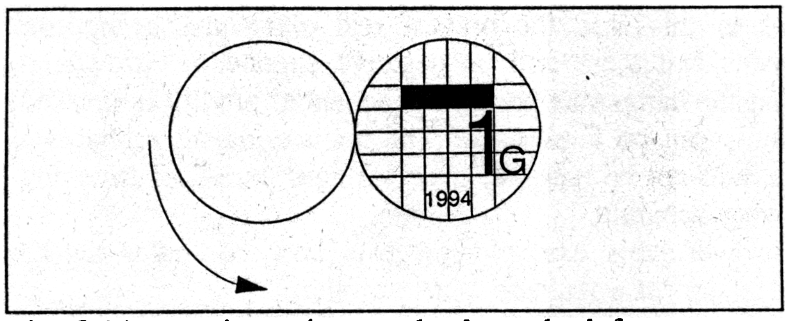

Fig. 8 The moving coin completely at the left

Now comes the mathematization phase. We add axes to the figure, so that the centres are on the x-axis and the origin coincides with the contact point in the starting situation. The fixed point P on the right coin is at the origin at the start. The scales of the two axes are both chosen to be equal to the radius of the guilder.

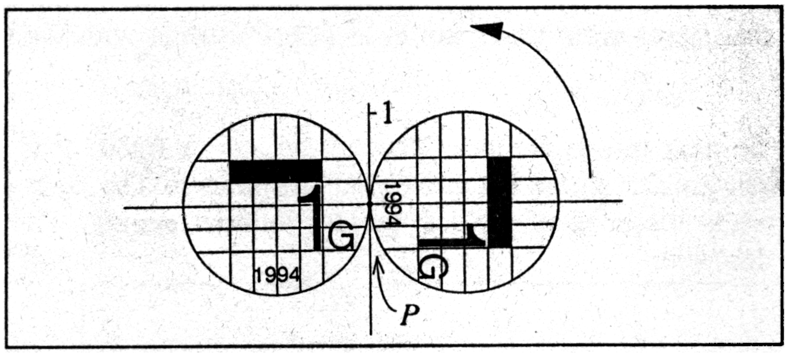

Fig. 9 Adding axes

In order to find the motion equations for P, we decompose the movement into two components. The first is the rotation of the centre of the right coin around the centre of the left one, and the second is the movement of P around the centre of the rotating coin.
The equations are, respectively:

$x_1(t) = 2\cos(t) - 1$
$y_1(t) = 2\sin(t)$

and

$x_2(t) = \cos(2t+\pi)$
$y_2(t) = \sin(2t+\pi)$

Addition of these components gives the complete movement of P:

$x(t) = 2\cos(t) - 1 + \cos(2t+\pi)$
$y(t) = 2\sin(t) + \sin(2t+\pi)$

So far, computer algebra was not of any help. From now on Derive of course will be very valuable.

```
#1:  X1(t) := 2·COS(t) - 1
#2:  Y1(t) := 2·SIN(t)
#3:  [X1(t), Y1(t)]
#4:  ""
#5:  X2(t) := COS(2·t + π)
#6:  Y2(t) := SIN(2·t + π)
#7:  [X2(t), Y2(t)]
#8:  ""
#9:  X(t) := X1(t) + X2(t)
#10: Y(t) := Y1(t) + Y2(t)
#11: [X(t), Y(t)]
#12:
```

Fig. 10 Algebra-window

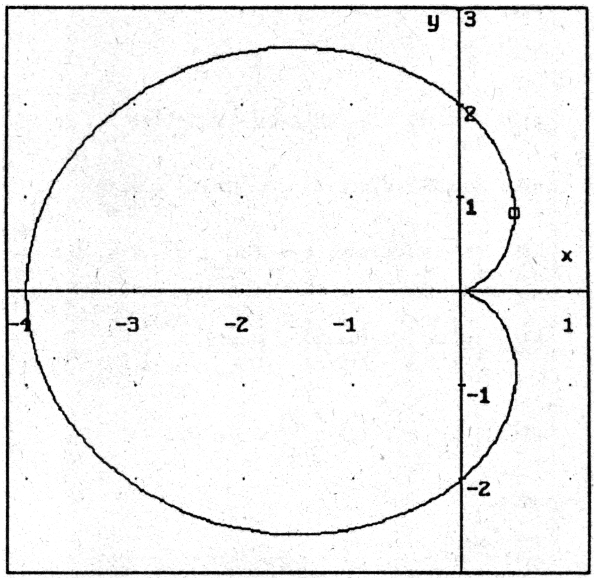

Fig. 11 Plot-window

The graph of the cardioid forms the answer to the original problem, but in its turn it is a source for further investigations. For example, what are the possible values of the first coordinate of the points of this cardioid? Tracing the curve (which also gives a good impression of the motion speed), we find a maximum value of x that seems to be about ½. That is an interesting outcome indeed. Is this just the result of a numerical roundoff error, or is the maximum distance from P to the y-axis exactly half a radius? Here we go back from the graphics to the algebra, and again we use Derive to find the extreme values of $t \to x(t)$. Figure 12 shows the results.

#11: $[X(t),\ Y(t)]$

#12: ""

#13: $\dfrac{d}{dt}\ X(t) = 2\cdot SIN(2\cdot t) - 2\cdot SIN(t)$

#14: SOLVE(2·SIN(2·t) - 2·SIN(t) = 0, t)

#15: $\left[t = 0,\ t = \pi,\ t = 2\cdot\pi,\ t = -2\cdot\pi,\ t = \dfrac{\pi}{3},\ t = -\dfrac{\pi}{3}\right]$

#16: $\left[X\left[\dfrac{\pi}{3}\right],\ Y\left[\dfrac{\pi}{3}\right]\right] = \left[\dfrac{1}{2},\ \dfrac{\sqrt{3}}{2}\right]$

#17: $\left[X1\left[\dfrac{\pi}{3}\right],\ Y1\left[\dfrac{\pi}{3}\right]\right] = [0,\ \sqrt{3}]$

#18: ""

#19:

Fig. 12 The Derive-screen for finding the maximum value of $x(t)$

Line 15 gives the zeros of the derivative of $x(t)$. In line 16 $t=\pi/3$ is substituted. The point of the cardioid that is most to the right thus is $(\frac{1}{2},\frac{1}{2}\sqrt{3})$.
Fine, the maximum value of x is exactly $\frac{1}{2}$. Because beautiful answers are suspect, we substituted $t=\pi/3$ in the equations of the circular movement that describes the centre of the rotating coin as well (see line 16). To our surprise, Derive's response shows that the centre at this moment is exactly on the y-axis. We now have discovered an interesting property of the cardioid: the point P reaches the extreme value of x when the centre of the rotating coin is on the y-axis.
Can we also see this geometrically? Of course we can. When P is in the position with the maximum x-coordinate, the speed of the movement in the x-direction equals 0. The speedvector is thus vertical. Because the speedvectors of the two component movements, $\vec{v}_1$ and $\vec{v}_2$, have equal lengths, they should obviously have an opposite angle with the y-axis in order to have a vertical sum. The figure 13 shows that this is indeed the case when M_2 is on the y-axis.

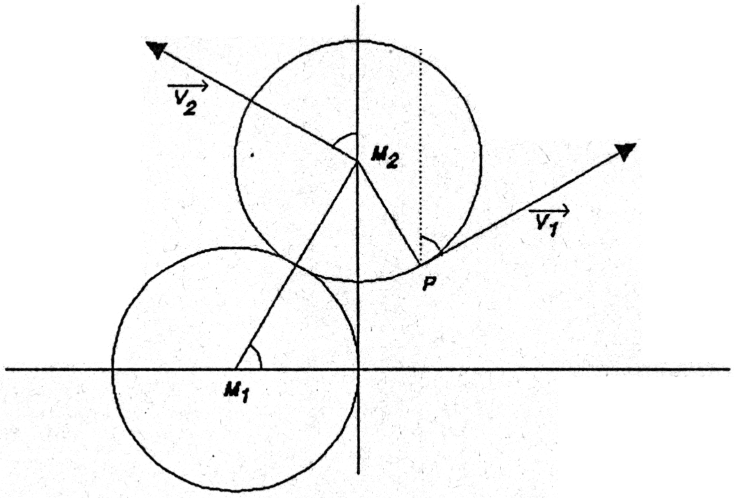

Fig. 13 The key to the geometric solution

Conclusion

Let us briefly look back at the five hypotheses that were stated at the beginning of this article to see if technology really added something in the presented examples.

1 Realistic contexts

The two examples indeed represent real-world problems. In the Chinese population problem in particular, the technology is useful in overcoming the calculational inconvenience that cause the dimensions of the matrix and thus makes the context more accessible.

2 Exploration

In the first example, computer algebra enables the student to calculate easily the future development over a few years. This forms a natural exploration phase, which would not have been possible without technology.
Once that the equations in the second example are set up, the graphical representation offers an excellent starting point for further investigations. The TRACE facility provides a way of exploring the speed of the movement.

3 Integration

In searching an eigenvalue of the population matrix, the algebraic method fails. The technology provides a graphical and a numerical alternative. From the combination of these different approaches, a more integrated view on mathematics will emerge..
In the second example, the graph is the starting point. From this picture a conjecture is made about the maximum value of x. This conjecture is proved algebraically. The geometric proof, at the end, is more convincing and elegant. Again, different solution strategies are integrated.

4 Dynamics

The investigation of the population development is dynamic in itself. The zooming in that takes place to obtain the approximated eigenvalue with the graphical method is another dynamical aspect. The effect of variations in the model, in this case the new policies, can easily be calculated, which is a third dynamic characteristic.
In the second example, we already mentioned the speed aspect using TRACE. More dynamics might be introduced when coins of different sizes would be considered. Actually, this is the way the situation is changed in the 'Movements in the plane' booklet.

5 Flexibility

The two examples both demand flexibility from the student. The ability to change the perspective and to combine different skills and techniques is clearly present.
In the first example, matrix algebra, graphs and numerical methods are combined. In the second one, the sequence graph-algebra-geometry is followed.
Of course, this flexibility is not easy to acquire for students. However, it is an important skill in mathematics and maybe in life in general!

Let us finish with an important remark. In the Netherlands, the classroom experience with computer algebra at upper secondary level is limited. Therefore, you did not find any observations from students' behaviour in this article. Much of what is stated is extrapolated from experiences with graphics calculators or with different student populations. This is, of course, an important restriction. If the five hypotheses that are stated in this article will turn out to hold for the use of computer algebra at upper secondary level in the Netherlands still is uncertain. Hopefully, future classroom experiments will clarify these points!

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